

**HAMILTON-JACOBI TREATMENT OF THE SCALAR FIELD
COUPLED TO TWO FERMIONS**

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Abstract

The constrained field system, the scalar field coupled to two flavours of fermions through Yukawa couplings, is treated by using the Hamilton-Jacobi approach. The equations of motion are obtained as total differential equations in many variables. The equations of motion are in exact agreement with those equations obtained using Dirac's method. Due to the Hamilton-Jacobi quantization, the path integral of the scalar field coupled to two fermions is obtained directly as an integration over the canonical phase space without using any gauge fixing condition.

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1 Introduction

Investigation the Hamiltonian treatment of constrained systems was initiated by Dirac [1], which is the most common method. Considering the primary constraints first is the main feature of his method. All constraints are obtained using consistency conditions. Besides, he showed that the number of degrees of freedom of the dynamical system can be reduced. Hence, the equations of motion of constrained system are obtained in terms of arbitrary parameters.

The canonical method, or Güler's method, developed Hamilton-Jacobi approach [2-3] to investigate constrained systems. In Refs. [4, 5], the equivalent Lagrangian method is used to obtain the set of Hamilton-Jacobi partial differential equations (HJPDE). In the Hamilton-Jacobi treatment of constrained systems [6-9]: (i) the distinction between first- and second-class constraints is unnecessary (ii) the equations of motion are obtained as total differential equations in many variables (iii) the investigation of the integrability conditions is required (iv) that gauge fixing is not necessary.

Quantization of the constrained Hamiltonian systems can be achieved by using several methods which are operator methods [1, 10], by path integral quantization [11-13], and based on the canonical method [14], which called the Hamilton-Jacobi quantization [15-20].

We investigate in the present work the scalar field coupled to two flavours of fermions through Yukawa couplings by using both Hamilton-Jacobi formulation and Dirac's method. Note that, Yukawa couplings are named after Hideki Yukawa interacted a scalar field φ with a Dirac field ψ of the type, which can be used to describe the strong nuclear force between nucleons, mediated by pions, and used in the Standard Model to describe the coupling between the Higgs field and massless quark and electron fields. Through spontaneous symmetry breaking, the fermions acquire a mass proportional to the vacuum expectation value of the Higgs field. The path integral quantization of the scalar field coupled to two flavours of fermions will be also quantize by using the Hamilton-Jacobi quantization.

In this paper both methods, Dirac's and Güler's are used to tackle the system of the scalar field coupled to two flavours of fermions through Yukawa couplings. This paper is arranged as follows: Hamilton-Jacobi approach is presented in Sec.2. Dirac's method is used in Sec.3. Hamilton-

Jacobi method is applied in sec.4 and the quantization of the system in Sec.5. The paper closes with a conclusion in Sec. 6.

2 Hamilton-Jacobi approach

The Hamilton-Jacobi approach can be introduced as follows:
If the rank of the Hess matrix is:

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (1)$$

is $(n - r)$, $r < n$, then the standard definition of a linear momenta is:

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (2)$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n, \quad (3)$$

enables us to solve Eq.(2) for $\dot{q}_a$ as

$$\dot{q}_b = \omega_b(q_i, \dot{q}_\mu, p_a). \quad (4)$$

Substituting Eq.(4), into Eq.(3), we obtain the constraints as

$$H'_\mu = p_\mu + H_\mu(\tau, q_i, p_a) = 0, \quad (5)$$

where

$$H_\mu = - \frac{\partial L}{\partial \dot{q}_\mu} \Big|_{\dot{q}_a \equiv \omega_a}. \quad (6)$$

The usual Hamiltonian H_0 is defined as:

$$H_0 = -L + p_a \dot{\omega}_a - \dot{q}_\mu H_\mu. \quad (7)$$

Like functions H_μ , the function H_0 is not an explicit function of the velocities $\dot{q}_\nu$. Therefore, the Hamilton-Jacobi function $S(\tau, q_i)$ should satisfy the following set of Hamilton-Jacobi partial differential equations (HJPDE) simultaneously for an extremum of the function:

$$H'_\alpha \left(t_\beta, q_\alpha, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial t_0} \right) = 0, \quad (8)$$

where

$\alpha, \beta = 0, n - r + 1, \dots, n;$ $a = 1, 2, \dots, n - r,$ and

$$H'_\alpha = p_\alpha + H_\alpha. \quad (9)$$

The canonical equations of motion are given as total differential equations in variables t_β ,

$$dq_p = \frac{\partial H'_\alpha}{\partial p_p} dt_\alpha, \quad p = 0, 1, \dots, n; \quad \alpha = 0, n - r + 1, \dots, n, \quad (10)$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad a = 1, \dots, n - r, \quad (11)$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \quad \alpha = 0, n - r + 1, \dots, n, \quad (12)$$

$$dZ = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha \right), \quad (13)$$

where

$$Z \equiv S(t_\alpha, q_a), \quad (14)$$

being the action. Thus, the analysis of a constrained system is reduced to solve equations (10-12) with constraints

$$H'_\alpha(t_\beta, q_a, P_i) = 0, \quad \alpha, \beta = 0, n - r + 1, \dots, n. \quad (15)$$

Since the equations above are total differential equations, integrability conditions should be checked. These equations of motion are integrable [3-5,8] if and only if the variations of H'_α vanish identically as:

$$dH'_\alpha = 0. \quad (16)$$

If they do not vanish identically, then we consider them as new constraints. This procedure is repeated until a complete system is obtained.

2.1 Hamilton-Jacobi Quantization

Path integral formulation of the constrained systems is studied in Refs. [14-20]. One has to consider a singular Lagrangian, as seen in previous section, for computing the Hamilton-Jacobi path integral. The canonical Hamiltonian H_0 defined in Eq. (6), and the set of HJPDE is expressed in Eq. (7). As we define

$$p_\beta = \frac{\partial S[q_a; x_\alpha]}{\partial x_\beta}, \quad (17)$$

and

$$p_a = \frac{\partial S[q_a; x_\alpha]}{\partial q_a}, \quad (18)$$

with $x_0 = t$ and S being the action. The total differential equations given in (10-13) are integrable if (16) are hold [21]. If conditions (16) are not satisfied identically, one considers them as new constraints and again consider their variations.

Thus, repeating this procedure one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide to obtain the set of canonical phase space coordinates (q_a, p_a) as functions of t_α , besides the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generator of canonical transformations given by parameters t_α . In this case path integral representation may be written as

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-p} dq^a dp^a \exp \left\{ \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha} \right) dt_\alpha \right\}, \quad (19)$$

$$a = 1, \dots, n-p, \quad \alpha = 0, n-p+1, \dots, n.$$

In fact, this path integral is an integration over the canonical phase-space coordinates (q^a, p^a) .

3 Dirac's method

In this section, we treat the constrained field system by Dirac's method [22].

Consider one loop order the self-energy for the scalar field φ , with a mass m , coupled to two flavours of fermions, with masses m_1 and m_2 , due to Yukawa couplings

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2 \varphi^2 - \frac{1}{6}\lambda \varphi^3 + \sum_i \bar{\psi}_{(i)}(i\gamma^\mu \partial_\mu - m_i)\psi_{(i)} - g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad \mu = 0, 1, 2, 3, \quad (20)$$

where λ is parameter and g constant, φ , $\psi_{(i)}$, and $\bar{\psi}_{(i)}$ are odd ones. We are adopting the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

The Lagrangian function (20) is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (21)$$

is one.

The generalized momenta (2,3)

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \partial^0 \varphi, \quad (22)$$

$$p_{(i)} = \frac{\partial L}{\partial \dot{\psi}_{(i)}} = i\bar{\psi}_{(i)}\gamma^0 = -H_{(i)}, \quad i = 1, 2, \quad (23)$$

$$\bar{p}_{(i)} = \frac{\partial L}{\partial \dot{\bar{\psi}}_{(i)}} = 0 = -\bar{H}_{(i)}. \quad (24)$$

Where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual $\psi_{(i)}$ as a column vector and $\bar{\psi}_{(i)}$ as a row vector implies that $p_{(i)}$ will be a row vector while $\bar{p}_{(i)}$ will be a column vector.

Since the rank of the Hess matrix is one, one may solve (22) for $\partial^0 \varphi$ as:

$$\partial^0 \varphi = p_\varphi \equiv \omega. \quad (25)$$

The usual Hamiltonian H_0 is given as:

$$H_0 = -L + \omega p_\varphi + \partial_0 \psi_{(i)} p_{(i)} \Big|_{p_{(i)} = -H_{(i)}} + \partial_0 \bar{\psi}_{(i)} \bar{p}_{(i)} \Big|_{\bar{p}_{(i)} = -\bar{H}_{(i)}}, \quad (26)$$

or

$$H_0 = \frac{1}{2}(p_\varphi^2 - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} \\ + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad a = 1, 2, 3. \quad (27)$$

Eqs. (23), and (24) lead to the primary constraints

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i\bar{\psi}_{(i)}\gamma^0 = 0, \quad (28)$$

and

$$\bar{H}'_{(i)} = \bar{p}_{(i)} + \bar{H}_{(i)} = \bar{p}_{(i)} = 0, \quad (29)$$

respectively. These constraints lead to the total Hamiltonian

$$H_T = H_0 + \lambda_{(i)}H'_{(i)} + \bar{\lambda}_{(i)}\bar{H}'_{(i)}, \quad (30)$$

or

$$H_T = \frac{1}{2}(p_\varphi^2 - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} \\ + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) + \lambda_{(i)}(p_{(i)} - i\bar{\psi}_{(i)}\gamma^0) + \bar{\lambda}_{(i)}\bar{p}_{(i)}. \quad (31)$$

According to Dirac's method, the time derivative of the primary constraints should be zero, that is

$$\dot{H}'_{(1)} = \{H'_{(1)}, H_T\} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi\bar{\psi}_{(2)} - i\bar{\lambda}_{(1)}\gamma^0 \approx 0, \quad (32)$$

$$\dot{H}'_{(2)} = \{H'_{(2)}, H_T\} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi\bar{\psi}_{(1)} - i\bar{\lambda}_{(2)}\gamma^0 \approx 0, \quad (33)$$

$$\dot{\bar{H}}'_{(1)} = \{\bar{H}'_{(1)}, H_T\} = -(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)} - i\gamma^0 \lambda_{(1)} \approx 0, \quad (34)$$

$$\dot{\bar{H}}'_{(2)} = \{\bar{H}'_{(2)}, H_T\} = -(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} - i\gamma^0 \lambda_{(2)} \approx 0. \quad (35)$$

Eqs. (32-35) fix the multipliers $\bar{\lambda}_{(1)}$, $\bar{\lambda}_{(2)}$, $\lambda_{(1)}$, and $\lambda_{(2)}$ respectively as

$$i\bar{\lambda}_{(1)}\gamma^0 = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi\bar{\psi}_{(2)}, \quad (36)$$

$$i\bar{\lambda}_{(2)}\gamma^0 = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi\bar{\psi}_{(1)}, \quad (37)$$

$$i\gamma^0 \lambda_{(1)} = -(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi \psi_{(2)}, \quad (38)$$

$$i\gamma^0 \lambda_{(2)} = -(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi \psi_{(1)}. \quad (39)$$

Multiplying Eqs. (36) and (37) from the right and Eqs. (38) and (39) from the left by $-i\gamma^0$, we obtain:

$$\bar{\lambda}_{(1)} = \bar{\psi}_{(1)} (\overleftarrow{\partial}_a \gamma^a - im_1) \gamma^0 - ig\varphi \bar{\psi}_{(2)} \gamma^0, \quad (40)$$

$$\bar{\lambda}_{(2)} = \bar{\psi}_{(2)} (\overleftarrow{\partial}_a \gamma^a - im_2) \gamma^0 - ig\varphi \bar{\psi}_{(1)} \gamma^0, \quad (41)$$

$$\lambda_{(1)} = -\gamma^0 (\gamma^a \partial_a + im_1)\psi_{(1)} - ig\varphi \gamma^0 \psi_{(2)}, \quad (42)$$

$$\lambda_{(2)} = -\gamma^0 (\gamma^a \partial_a + im_2)\psi_{(2)} - ig\varphi \gamma^0 \psi_{(1)}. \quad (43)$$

There are no secondary constraints. Taking suitable linear combinations of constraints, one has to find all numbers of second-class ones, there are

$$\Phi_i = H'_{(i)} = p_{(i)} - i\bar{\psi}_{(i)} \gamma^0, \quad i = 1, 2, \quad (44)$$

$$\Phi_3 = \bar{H}'_{(1)} = \bar{p}_{(1)}, \quad (45)$$

and

$$\Phi_4 = \bar{H}'_{(2)} = \bar{p}_{(2)}. \quad (46)$$

The equations of motion are read as

$$\dot{\varphi} = \{\varphi, H_T\} = p_\varphi, \quad (47)$$

$$\dot{\psi}_{(i)} = \{\psi_{(i)}, H_T\} = \lambda_{(i)}, \quad (48)$$

$$\dot{\bar{\psi}}_{(i)} = \{\bar{\psi}_{(i)}, H_T\} = \bar{\lambda}_{(i)}, \quad (49)$$

$$\dot{p}_\varphi = \{p_\varphi, H_T\} = m^2\varphi + \frac{1}{2}\lambda\varphi^2 + g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad (50)$$

$$\dot{p}_{(1)} = \{p_{(1)}, H_T\} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\varphi\bar{\psi}_{(2)}, \quad (51)$$

$$\dot{p}_{(2)} = \{p_{(2)}, H_T\} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) + g\varphi\bar{\psi}_{(1)}, \quad (52)$$

$$\dot{\bar{p}}_{(1)} = \{\bar{p}_{(1)}, H_T\} = -(i\gamma^a\partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)} - i\gamma^0\lambda_{(1)}, \quad (53)$$

$$\dot{\bar{p}}_{(2)} = \{\bar{p}_{(2)}, H_T\} = -(i\gamma^a\partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} - i\gamma^0\lambda_{(2)}. \quad (54)$$

Differentiate Eq. (47) with respect to time, and substituting from Eq. (50), we get:

$$\ddot{\varphi} - m^2\varphi - \frac{1}{2}\lambda\varphi^2 - g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) = 0. \quad (55)$$

Substituting from Eqs. (42) and (43) into Eqs. (48), (53) and (54), we get:

$$(i\gamma^\mu\partial_\mu - m_1)\psi_{(1)} - g\varphi\psi_{(2)} = 0, \quad (56)$$

$$(i\gamma^\mu\partial_\mu - m_2)\psi_{(2)} - g\varphi\psi_{(1)} = 0, \quad (57)$$

$$\dot{\bar{p}}_{(i)} = 0, \quad i = 1, 2. \quad (58)$$

From Eqs. (40) and (41) into Eq. (49), we have

$$\partial_0\bar{\psi}_{(1)} - i\gamma^0\bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) - g\varphi\bar{\psi}_{(2)} = 0, \quad (59)$$

$$\partial_0\bar{\psi}_{(2)} - i\gamma^0\bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) - g\varphi\bar{\psi}_{(1)} = 0. \quad (60)$$

In the following section the same system will be discussed by using Hamilton-Jacobi approach.

4 Hamilton-Jacobi method

In this section, the constrained system is going now to be tackled by using Hamilton-Jacobi treatment [23], which solve the gauge fixing problem naturally.

The set of (HJPDE) (8) read as

$$H'_0 = p_0 + H_0 = p_0 + \frac{1}{2}(p^2_\varphi - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad (61)$$

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i\bar{\psi}_{(i)}\gamma^0 = 0, \quad (62)$$

$$\bar{H}'_{(i)} = \bar{p}_{(i)} + \bar{H}_{(i)} = \bar{p}_{(i)} = 0. \quad (63)$$

Therefore, the total differential equations for the characteristic (10), (11) and (12) are:

$$d\varphi = p_\varphi d\tau, \quad (64)$$

$$d\psi_{(i)} = d\psi_{(i)}, \quad (65)$$

$$d\bar{\psi}_{(i)} = d\bar{\psi}_{(i)}, \quad (66)$$

$$dp_\varphi = \left[m^2 \varphi + \frac{1}{2}\lambda \varphi^2 + g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) \right] d\tau, \quad (67)$$

$$dp_{(1)} = \left[\bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi \bar{\psi}_{(2)} \right] d\tau, \quad (68)$$

$$dp_{(2)} = \left[\bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi \bar{\psi}_{(1)} \right] d\tau, \quad (69)$$

$$d\bar{p}_{(1)} = \left[-(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi \psi_{(2)} \right] d\tau - i\gamma^0 d\psi_{(1)}, \quad (70)$$

$$d\bar{p}_{(2)} = \left[- (i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} \right] d\tau - i\gamma^0 d\psi_{(2)}. \quad (71)$$

The integrability conditions ($dH'_\alpha = 0$) imply that the variation of the constraints $H'_{(i)}$ and $\bar{H}'_{(i)}$ should be identically zero, that is

$$dH'_{(i)} = dp_{(i)} - i d\bar{\psi}_{(i)} \gamma^0 = 0, \quad (72)$$

$$d\bar{H}'_{(i)} = d\bar{p}_{(i)} = 0. \quad (73)$$

The following equations of motion:
From Eq. (64), we obtain

$$\dot{\varphi} = p_\varphi. \quad (74)$$

Substituting from Eqs. (68) and (69) into Eq. (72), we get

$$i\partial_0\bar{\psi}_{(1)}\gamma^0 - \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) - g\varphi\bar{\psi}_{(2)} = 0, \quad (75)$$

$$i\partial_0\bar{\psi}_{(2)}\gamma^0 - \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) - g\varphi\bar{\psi}_{(1)} = 0. \quad (76)$$

Substituting from Eqs. (70) and (71) into Eq. (73), we obtain

$$(i\gamma^\mu\partial_\mu - m_1)\psi_{(1)} - g\varphi\psi_{(2)} = 0, \quad (77)$$

$$(i\gamma^\mu\partial_\mu - m_2)\psi_{(2)} - g\varphi\psi_{(1)} = 0. \quad (78)$$

One notes that the integrability conditions are not identically zero, they are added to the set of equations of motion.

From Eqs. (67-69), we get the following equations of motion:

$$\dot{p}_\varphi = m^2\varphi + \frac{1}{2}\lambda\varphi^2 + g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad (79)$$

$$\dot{p}_{(1)} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\varphi\bar{\psi}_{(2)}, \quad (80)$$

$$\dot{p}_{(2)} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) + g\varphi\bar{\psi}_{(1)}. \quad (81)$$

Substituting from Eqs. (77) and (78) into (70) and (71), we get

$$\dot{\bar{p}}_{(i)} = 0, \quad i = 1, 2. \quad (82)$$

Differentiate Eq. (74) with respect to time, and making use of (79), we obtain

$$\ddot{\varphi} - m^2 \varphi - \frac{1}{2} \lambda \varphi^2 - g(\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) = 0. \quad (83)$$

5 Hamilton-Jacobi quantization

In this section, we use the Hamilton-Jacobi quantization to obtain the path integral quantization of the scalar field coupled to two flavours of fermions through Yukawa couplings.

First, one must check whether the set of equations (64-71) is integrable or not. So, one has to consider the total variations of the constraints. In fact

$$dH'_{(i)} = dp_{(i)} - i d\bar{\psi}_{(i)} \gamma^0 = 0, \quad (84)$$

$$d\bar{H}'_{(i)} = d\bar{p}_{(i)} = 0. \quad (85)$$

The constraints (62) and (63), lead us to obtain $d\bar{\psi}_{(i)}$ and $d\psi_{(i)}$ in terms of dt

$$d\bar{\psi}_{(1)} i\gamma^0 = [\bar{\psi}_{(1)} (i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi \bar{\psi}_{(2)}] dt, \quad (86)$$

$$d\bar{\psi}_{(2)} i\gamma^0 = [\bar{\psi}_{(2)} (i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi \bar{\psi}_{(1)}] dt, \quad (87)$$

$$i\gamma^0 d\psi_{(1)} = [-(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi \psi_{(2)}] dt, \quad (88)$$

and

$$i\gamma^0 d\psi_{(2)} = [-(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi \psi_{(1)}] dt. \quad (89)$$

We obtain that the set of equations (64-71) is integrable. Making use of (13), and (61-63), we can write the canonical action integral as

$$Z = \int d^4x \left[\frac{1}{2} (p_\varphi^2 + \partial_a \varphi \partial^a \varphi) - \frac{1}{2} m^2 \varphi^2 - \frac{1}{6} \lambda \varphi^3 + \bar{\psi}_{(i)} (i \gamma^\mu \partial_\mu - m_i) \psi_{(i)} - g \varphi (\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) \right], \quad (90)$$

Now the path integral representation (19) is given by

$$\langle out | S | In \rangle = \int \prod_i^2 d\varphi dp_\varphi d\psi_{(i)} d\bar{\psi}_{(i)} \exp \left\{ i \left[\int d^4x \frac{1}{2} (p_\varphi^2 + \partial_a \varphi \partial^a \varphi) - \frac{1}{2} m^2 \varphi^2 - \frac{1}{6} \lambda \varphi^3 + \bar{\psi}_{(i)} (i \gamma^\mu \partial_\mu - m_i) \psi_{(i)} - g \varphi (\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) \right] \right\}. \quad (91)$$

6 Conclusion

The scalar field coupled to two flavours of fermions through Yukawa couplings is discussed as constrained system [7] by using both Dirac's and Hamilton-Jacobi methods. In Dirac's method the total Hamiltonian composed by adding the constraints multiplied by Lagrange multipliers to the canonical Hamiltonian. In order to derive the equations of motion, one needs to redefine these unknown multipliers in an arbitrary way. However, in the Hamilton-Jacobi approach (or Güler's method)[2-8], there is no need to introduce Lagrange multipliers to the canonical Hamiltonian, then the Hamilton-Jacobi is simpler and more economical.

In Hamilton-Jacobi approach it is not necessary to distinguish between first-class and second-class constraints. Hamilton-Jacobi approach always in exact agreement with Dirac's method. Both the consistency conditions and the integrability conditions lead to the same constraints. Also the equations of motion in both approaches are the same.

Hamilton-Jacobi quantization is applied to the constrained field system with finite degrees of freedom by investigating the integrability conditions without using any gauge fixing condition.

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